Formal definition of a system of push down automata

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We define a transducer pushdown automaton (TPA) to be a finite automaton that contains an unbounded stack and an unbounded output tape in addition to its finite input tape. More formally, a TPA M_0 is a tuple $M_0 = (K_0, \Sigma_0, \Gamma_0, \Delta_0, s_0, F_0)$ where K_0 is a finite set of states, Σ_0 and Γ_0 are finite sets of symbols (the input and output alphabet, and the stack alphabet, respectively), Δ_0 is the transition function, $s_0 \in K_0$ is the start state, and $F_0 \subseteq K_0$ is the set of final states. The transition function Δ_0 is of the form $\Delta_0 \subseteq (K_0 \times \Sigma_0 \times (\Gamma_0 \cup \{\varepsilon\})) \to (K_0 \times (\Gamma_0 \cup \{\varepsilon\}) \times \Sigma_0^*)$, where ε is the empty word. Each transition consists of M_0 reading exactly one input symbol, popping at most one symbol from the stack, changing state, pushing at most one symbol to the stack, and appending a string to the output tape. Each configuration of M_0 is an element of $K_0 \times \Sigma_0^* \times \Gamma_0^* \times \Sigma_0^*$, consisting of the current state, the remaining symbols on the input tape, the contents of the stack, and the contents of the output tape. A transition exists between configuration $C = (\alpha, aw, bs, o)$ and configuration $C' = (\beta, w, cs, od)$ if there exists a rule $(\alpha, a, b) \to (\beta, c, d) \in \Delta_0$, where $\alpha, \beta \in K_0$, $a \in \Sigma_0$, $w \in \Sigma_0^*$, $b, c \in (\Gamma_0 \cup \{\varepsilon\})$, $s \in \Gamma_0^*$, $o, d \in \Sigma_0^*$. It can be seen that each TPA is deterministic and always halts.

We define $M = (K, \Sigma, \Gamma, \Delta, S, F)$ as a system of n TPAs, where each TPA i has a unique finite set of K_i states such that

$$K = \bigcup_{i=0}^{n-1} K_i, \bigcap_{i=0}^{n-1} K_i = \emptyset\,,$$

Σ is a finite alphabet of input/output symbols, $\# \ni \Sigma$, Γ is a finite alphabet of stack symbols, $\# \ni \Gamma$, $S = \{s_i : s_i \in K_i, 0 \le i < n\}$ contains the start state for each TPA i, and F contains the final states for each TPA i such that

$$F = \bigcup_{i=0}^{n-1} F_i, F_i \subseteq K_i\,.$$

The transition function, $\Delta \subseteq \Delta' \cup \Delta''$, is defined as a subset of the union of the sets of all intra-TPA transitions and inter-TPA transitions. The set of all intra-TPA transitions is

$$\Delta' = \bigcup_{i=0}^{n-1} ((K_i \times \Sigma \times (\Gamma \cup \{\varepsilon\})) \to (K_i \times (\Gamma \cup \{\varepsilon\}) \times \Sigma^*)) .$$

The set of all inter-TPA transitions is

$$\Delta'' = \bigcup_{i=0}^{n-2} ((F_i \times \{\#\} \times \{\varepsilon\}) \to (\{s_{i+1}\} \times \{\varepsilon\} \times \{\#\})) .$$

A configuration of M is an element of $K \times (\Sigma_{\#} \times \Gamma^* \times \Sigma_{\#})^n$ where $\Sigma_{\#} = (\Sigma^* \cup \{\#\})$. The initial configuration of M is

$$(s_0, (w\#, \varepsilon, \varepsilon), (\varepsilon, \varepsilon, \varepsilon), \ldots, (\varepsilon, \varepsilon, \varepsilon)) ,$$

where $w \in \Sigma^*$ is the input to M. A final configuration of M is of the form

$$(f, (\varepsilon, \gamma_0, \varepsilon), (\varepsilon, \gamma_1, \varepsilon), \ldots, (\#, \gamma_{n-1}, r)) ,$$

where $f \in F_{n-1}$, $r \in \Sigma^*$, and $\gamma_i \in \Gamma^*, 0 \leq i < n$.

Let '$\vdash$' be a binary relation on configurations called the transition. A transition exists between configuration C_1 and configuration C_2, denoted $C_1 \vdash C_2$, if C_1 is of the form

$$(\alpha, (\varphi, (aw\#, bs, o), \chi)) ,$$

where $\alpha \in K_i, w, o \in \Sigma^*, s \in \Gamma^*, a \in \Sigma, b \in (\Gamma \cup \{\varepsilon\}), \varphi \in (\Sigma_{\#} \times \Gamma^* \times \Sigma_{\#})^p, \chi \in (\Sigma_{\#} \times \Gamma^* \times \Sigma_{\#})^q, p + q + 1 = n$ and C_2 is of the form

$$(\beta, (\varphi, (w\#, cs, od), \chi)) ,$$

where $\beta \in K_i, d \in \Sigma^*, c \in (\Gamma \cup \{\varepsilon\})$ and a transition rule of the following form exists in Δ

$$(\alpha, a, b) \to (\beta, c, d) ,$$

or, if C_1 is of the form

$$(f, (\psi, (\#, \gamma_{p+1}, r), (\varepsilon, \varepsilon, \varepsilon), \omega)) ,$$

where $f \in F_i, \psi \in (\Sigma_{\#} \times \Gamma^* \times \Sigma_{\#})^p, \omega \in (\Sigma_{\#} \times \Gamma^* \times \Sigma_{\#})^q, p + q + 2 = n, \gamma_{p+1} \in \Gamma^*, r \in \Sigma^*$ and C_2 is of the form

$$(s_{i+1}, (\psi(\varepsilon, \gamma_{p+1}, \varepsilon), (r\#, \varepsilon, \varepsilon)\omega)) ,$$

where a transition rule of the following form exists in Δ

$$(f, \#, \varepsilon) \to (s_{i+1}, \varepsilon, \#)) .$$

We denote the reflexive and transitive closure of $\vdash$ as $\vdash^*$. An accepting computation for M with input w exists if and only if $C_{initial} \vdash^* C_{final}$ where $C_{initial}$ is an initial conifguration and C_{final} is a final configuration.