Complexity of continuous space machine operations

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Abstract. We investigate the computational complexity of an optical model of computation called the continuous space machine (CSM). We characterise worst case resource growth over time for each of the CSM's ten operations with respect to seven resource measures. Many operations exhibit unreasonably large growth rates thus motivating restrictions on the CSM, in particular we give a restriction called the $\mathcal{C}_2$-CSM.

1 Introduction

The computational model we study is relatively new and is called the continuous space machine (CSM) [11, 12, 13, 18, 19]. The CSM is inspired by classical Fourier optics and uses complex-valued images, arranged in a grid structure, for data storage. The program also resides in images. The CSM has the ability to perform Fourier transformation, complex conjugation, multiplication, addition, thresholding and resizing of images. It has simple control flow operations and is deterministic. To analyse such a model we define a total of seven complexity measures inspired by real-world resources. For example, spatial resolution corresponds to number of pixels.

A variant of the model with real inputs was previously shown [19] to decide the membership problem for all recursively enumerable languages, and as such is unreasonable in terms of implementation. Here, we build on this work by showing the growth in resource usage for each CSM operation. This leads to a restriction of the CSM that is more suited to the standard tools from analysis of algorithms and complexity theory.

2 The CSM

We begin by informally describing the model, this brief overview is not intended to be complete: Detailed definitions and discussions can be found in [18, 19].

Definition 1 (complex-valued image). *A complex-valued image (or simply, image) is a function* $f : [0, 1) \times [0, 1) \to \mathbb{C}$*, where* $[0, 1)$ *is the half-open real unit interval.*

We let $\mathcal{I}$ denote the set of all complex-valued images. $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ and $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$. For a given CSM M we let $\mathcal{N}$ be a countable set of images that encode M's addresses. Also for a given M there is an address encoding function $\mathfrak{E} : \mathbb{N} \to \mathcal{N}$ such that $\mathfrak{E}$ is Turing machine decidable, under some *reasonable*[3] representation of images as words. An address is simply an element of $\mathbb{N} \times \mathbb{N}$.

Definition 2 (CSM). *A CSM is a quintuple $M = (\mathfrak{E}, L, I, P, O)$, where*

$\mathfrak{E} : \mathbb{N} \to \mathcal{N}$ *is the address encoding function*
$L = ((s_\xi, s_\eta), (a_\xi, a_\eta), (b_\xi, b_\eta))$ *are the addresses: sta, a, and b,*
$I = ((\iota_{1_\xi}, \iota_{1_\eta}), \ldots, (\iota_{k_\xi}, \iota_{k_\eta}))$ *are the addresses of the k input images,*
$P = \{(\zeta_1, p_{1_\xi}, p_{1_\eta}), \ldots, (\zeta_r, p_{r_\xi}, p_{r_\eta})\}$ *are the r programming symbols and their addresses where* $\zeta_j \in (\{h, v, *, \cdot, +, \rho, st, ld, br, hlt\} \cup \mathcal{N}) \subset \mathcal{I}$,
$O = ((o_{1_\xi}, o_{1_\eta}), \ldots, (o_{l_\xi}, o_{l_\eta}))$ *are the addresses of the l output images.*

Each address is an element from $\{0, 1, \ldots, \Xi - 1\} \times \{0, 1, \ldots, \mho - 1\}$ *where* $\Xi, \mho \in \mathbb{N}^+$. *Addresses a and b are distinct.*

Addresses whose contents are not specified by P in a CSM definition are assumed to contain the constant image $f(x, y) = 0$. We interpret this definition to mean that M is (initially) defined on a grid of images bounded by the constants Ξ and $\mho$, in the horizontal and vertical directions respectively.

In our grid notation the first and second elements of an address tuple refer to the horizontal and vertical axes of the grid respectively, and image $(0, 0)$ is at the bottom left-hand corner of the grid. The images in a grid have the same orientation as the grid. Fig. 1 gives the CSM operations in this grid notation. Configurations are defined in a straightforward way as a tuple $\langle c, e \rangle$ where c is an address called the control and e represents the grid contents. In the sequel we write $\widehat{c}$ to mean the image (or instruction) at address c. It is beyond the scope of this paper to give CSM algorithms and so this informal description is sufficient for our analysis in Sect. 4. For a more thourough introduction see [18, 19].

3 Complexity measures

We want our complexity measures to be straightforward to analyse, while at the same time to be meaningful by reflecting the reality of optical computing. All finite resource bounding functions are from $\mathbb{N}$ into $\mathbb{N}$ and have the usual properties [1].

Definition 3. *The* TIME *complexity of a CSM M is the number of configurations in the computation sequence of M, beginning with the initial configuration and ending with the first final configuration.*

Definition 4. *The* GRID *complexity of a CSM M is the minimum number of images, arranged in a rectangular grid, for M to compute correctly on all inputs.*

[3] Other authors have also raised this representation issue for different models, but with similar motivations. See [18] for further discussion.

$\boxed{\mathrm{h}}$	: replace image in a with its horizontal 1D Fourier transform (FT).
$\boxed{\mathrm{v}}$	: replace image in a with its vertical 1D FT.
$\boxed{*}$	: replace image in a with its complex conjugate.
$\boxed{\cdot}$	: multiply (point by point) the two images in a and b. Store result in a.
$\boxed{+}$	: perform a complex addition of a and b. Store result in a.
$\boxed{\rho}\boxed{z_{\mathrm{l}}}\boxed{z_{\mathrm{u}}}$	: $z_{\mathrm{l}}, z_{\mathrm{u}} \in \mathcal{I}$; filter the image in a by amplitude using z_{l} and z_{u} as lower and upper amplitude threshold images, respectively.
$\boxed{\mathrm{st}}\boxed{\xi_1}\boxed{\xi_2}\boxed{\eta_1}\boxed{\eta_2}$	: $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{N}$; copy the image in a into the rectangle of images whose bottom left-hand corner address is (ξ_1, η_1) and whose top right-hand corner address is (ξ_2, η_2).
$\boxed{\mathrm{ld}}\boxed{\xi_1}\boxed{\xi_2}\boxed{\eta_1}\boxed{\eta_2}$	: $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{N}$; copy into a the rectangle of images whose bottom left-hand corner address is (ξ_1, η_1) and top right-hand corner address is (ξ_2, η_2).
$\boxed{\mathrm{br}}\boxed{\xi}\boxed{\eta}$	: $\xi, \eta \in \mathbb{N}$; unconditionally branch to the image at address (ξ, η).
$\boxed{\mathrm{hlt}}$	: halt.

Fig. 1. The set of CSM operations, given in our informal grid notation.

From the CSM definition GRID is at least $\Xi\,\mathcal{Y}$. In previous work [11, 12, 13, 19] the number of grid images remained constant throughout a computation. Here we alter the CSM (by introducing the address encoding function $\mathfrak{E}$) so that GRID may grow over time.

Next we define SPATIALRES. Let a *pixel* λ be a constant complex function defined on a real-valued rectangle with rational endpoints, $\lambda : [0, W) \times [0, H) \to z$ where $z \in \mathbb{C}$; $W, H \in \mathbb{Q}$; $0 < W, H \leqslant 1$; and $[0, W), [0, H) \subset \mathbb{R}$. A *raster image* is an image composed entirely of nonoverlapping pixels, each pixel is of equal height H, equal width W, identical orientation, and arranged into Φ columns and Ψ rows where $\Phi W = 1 = \Psi H$. Let the *spatial resolution* of a raster image be $\Phi\Psi$. Let $S : \mathcal{I} \times (\mathbb{N} \times \mathbb{N}) \to \mathcal{I}$, where $S(f(x, y), (\Phi, \Psi))$ is a raster image, with $\Phi\Psi$ pixels arranged in Φ columns and Ψ rows, that approximates $f(x, y)$. If we choose a reasonable and realistic S then the details of S are not important.

Definition 5. *The* SPATIALRES *complexity of a CSM M is the minimum $\Phi\Psi$ such that if each image $f(x, y)$ in the computation of M is replaced with $S(f(x, y), (\Phi, \Psi))$ then M computes correctly on all inputs.*

If no such $\Phi\Psi$ exists then M has infinite SPATIALRES complexity.

For AMPLRES complexity consider the function $A : \mathcal{I} \times \mathbb{N}^+ \to \mathcal{I}$,

$$A(f(x, y), \mu) = \left\lfloor |f(x, y)|\mu + \frac{1}{2} \right\rfloor \frac{1}{\mu} \exp(\mathrm{i} \times \arg(f(x, y)))\,, \tag{1}$$

where $|\cdot|$ gives the amplitudes of its image argument, $\arg(\cdot)$ gives the phase angles in the range $(0, 2\pi]$, and the floor operation operates separately on each image value. The value μ is the cardinality of the set of discrete nonzero amplitude values that each complex value in $A(f, \mu)$ can take, per half-open unit interval of amplitude. To aid in the understanding of Equation (1), recall that $f(x, y) = |f(x, y)| \exp(\mathrm{i} \times \arg(f(x, y)))$.

Definition 6. *The* AMPLRES *complexity of a CSM M is the minimum μ such that if each image $f(x, y)$ in the computation of M is replaced by $A(f(x, y), \mu)$ then M computes correctly on all inputs.*

If no such μ exists then M has infinite AMPLRES complexity.

Consider the function $P : \mathcal{I} \times \mathbb{N}^+ \to \mathcal{I}$ defined as

$$P(f(x, y), \mu) = |f(x, y)| \exp\left(\mathrm{i} \left\lfloor \arg(f(x, y)) \frac{\mu}{2\pi} + \frac{1}{2} \right\rfloor \frac{2\pi}{\mu}\right) . \quad (2)$$

The value μ is the cardinality of the set of discrete phase values that each complex value in $P(f, \mu)$ can take.

Definition 7. *The* PHASERES *complexity of a CSM M is the minimum μ such that if each image $f(x, y)$ in the computation of M is replaced by $P(f(x, y), \mu)$ then M computes correctly on all inputs.*

If no such μ exists then M has infinite PHASERES complexity.

Definition 8. *The* DYRANGE *complexity of a CSM M is the ceiling of the maximum of all the amplitude values stored in all of M's images during M's computation.*

Definition 9. *The* FREQ *complexity of a CSM M is the minimum optical frequency such that M computes correctly on all inputs.*

The concept of minimum optical frequency is explained in [19]. If approximations of a FT are sufficient for M, or if M does not execute h nor v, then M requires finite FREQ. If the original (unbounded) definitions of h and v must hold then M requires infinite FREQ. Using the traditional optical methods, any lower bound on SPATIALRES will impose a lower bound on FREQ [19], we should be aware of this in our complexity analysis.

Often we wish to make analogies between space on some well-known model and 'space-like' resources on the CSM. For this purpose we define the following convenient term.

Definition 10. *The* SPACE *complexity of a CSM M is the product of all of M's complexity measures except* TIME.

We argue that this definition is reasonable as it gives an upper bound on the information (e.g. number of bits) stored throughout a CSM computation.

We have defined the complexity of a computation (sequence of configurations) for each measure. We extend this definition to the complexity of a (possibly non-final) configuration in the obvious way. Also, we sometimes talk about the complexity of an image, this is simply the complexity of the configuration that the image is in. A more detailed explanation of the complexity measures can be found in [19], including a discussion on defining energy of computations in terms of the above measures.

	GRID	SPATIALRES	AMPLRES	DYRANGE	PHASERES	FREQ
h	G_T	∞ (1)	∞ (1)	∞ (2)	∞ (1)	∞ (1)
v	G_T	∞ (1)	∞ (1)	∞ (2)	∞ (1)	∞ (1)
$*$	G_T	$R_{\mathrm{S},T}$	$R_{\mathrm{A},T}$	$R_{\mathrm{D},T}$	$R_{\mathrm{P},T}$ (3)	ν_T
$\cdot$	G_T	$R_{\mathrm{S},T}$	$(R_{\mathrm{A},T})^2$ (4)	$(R_{\mathrm{D},T})^2$ (5)	$R_{\mathrm{P},T}$ (6)	ν_T
$+$	G_T	$R_{\mathrm{S},T}$	∞ (7)	$2R_{\mathrm{D},T}$ (8)	∞ (9)	ν_T
ρ	unbounded (10)	$R_{\mathrm{S},T}$	$R_{\mathrm{A},T}$	$R_{\mathrm{D},T}$	$R_{\mathrm{P},T}$	ν_T
st	unbounded (10)	$R_{\mathrm{S},T}$	$R_{\mathrm{A},T}$	$R_{\mathrm{D},T}$	$R_{\mathrm{P},T}$	ν_T
ld	unbounded (10)	unbounded (11)	$R_{\mathrm{A},T}$	$R_{\mathrm{D},T}$	$R_{\mathrm{P},T}$	unbounded (11)
br	G_T (12)	$R_{\mathrm{S},T}$	$R_{\mathrm{A},T}$	$R_{\mathrm{D},T}$	$R_{\mathrm{P},T}$	ν_T
hlt	G_T	$R_{\mathrm{S},T}$	$R_{\mathrm{A},T}$	$R_{\mathrm{D},T}$	$R_{\mathrm{P},T}$	ν_T

Table 1. CSM resource usage after one timestep. For each operation and complexity measure pair, the table entry defines the worst case upper bound on CSM resource usage at TIME $T+1$, in terms of resources used at TIME T: GRID $= G_T$, SPATIALRES $= R_{\mathrm{S},T}$, AMPLRES $= R_{\mathrm{A},T}$, DYRANGE $= R_{\mathrm{D},T}$, PHASERES $= R_{\mathrm{P},T}$ and FREQ $= \nu_T$. Theorems are cited in parentheses.

4 Worst case CSM resource usage

For the case of sequential computation it is usually obvious how the execution of a single operation will effect resource usage. In parallel models, execution of a single operation can lead to large growth in one timestep. For example a multiplication or shift operation in a unit cost parallel model (such as Pratt and Stockmeyer's unrestricted vector machines [16]) can double the length of a binary string in one step. When binary strings are interpreted as numbers, such multiplications and shifts quickly generate large values. Characterising resource growth is useful for proving upper bounds on power and setting model restrictions [1].

In this section we investigate the growth of complexity resources over TIME, with respect to CSM operations. We tackle this question for each operation and complexity measure pair. As expected, under certain operations some measures do not grow at all. Others grow at rates comparable to massively parallel models. By allowing operations like the FT we are mixing the continuous and discrete worlds, hence some measures grow to infinity in one timestep. This gives strong motivation for CSM restrictions and raises some interesting questions.

Table 1 summarises our results; the table defines the value of a complexity measure after execution of an operation (at TIME $T+1$). The complexity of a configuration at TIME $T+1$ is at least the value it was at TIME T, since complexity functions are nondecreasing. Our definition of TIME assigns unit time cost to each operation, hence we do not have a TIME column. Many entries are immediate from the definitions, otherwise a theorem is cited to the right of the entry.

In the sequel $\langle c, e\rangle_T$ is an an arbitrary CSM configuration at TIME T and $\widehat{c}$ is the instruction pointed to by the program control c. Also, G_T, R_{S_T}, R_{A_T}, R_{D_T}, R_{P_T} and ν_T are the GRID, SPATIALRES, AMPLRES, DYRANGE, PHASERES and FREQ respectively of configuration $\langle c, e\rangle_T$. Our resource growth analysis is worst case, hence we assume that at each computation step we want to preserve all information in each image (however for specific computations this may not be the case). Each of Theorems 1–12 trivially hold if the resources in question are infinite at TIME T, proofs are given only for the non-trivial finite case. We begin with resource usage after operations h or v for a number of complexity measures.

Theorem 1. *(h/v & SPATIALRES, AMPLRES, PHASERES and FREQ) Let either* $\widehat{c} = h$ *or* $\widehat{c} = v$. *Then* $R_{\mathrm{S}_{T+1}} = R_{\mathrm{A}_{T+1}} = R_{\mathrm{P}_{T+1}} = \nu_{T+1} = \infty$.

Proof. We give a proof for the non-trivial case where each measure is finite at TIME T. The statement is proved for the measure in question if there is no finite minimum value for that measure at TIME $T+1$. We use any rectangular step image, such as

$$a(x,y) = \begin{cases} \frac{1}{R_{\mathrm{A}_T}}, & \text{if } \frac{1}{2} - \frac{1}{R_{\mathrm{S}_{x}T}} \leqslant x < \frac{1}{2} \text{ and } \frac{1}{2} - \frac{1}{R_{\mathrm{S}_{y}T}} \leqslant y < \frac{1}{2}\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

$R_{\mathrm{S}_{x}T}$ and $R_{\mathrm{S}_{x}T}$ are the spatial resolutions in the horizontal and vertical directions respectively. The image $a(x,y)$ is representable with finite SPATIALRES, AMPLRES, PHASERES and FREQ. However its (horizontal or vertical) Fourier spectrum is a sinc function containing an infinite number of spatially separated components and is therefore not representable by finite SPATIALRES nor FREQ. The amplitudes of the peaks in this Fourier spectrum monotonically decrease in value, never reaching zero, and hence are not representable by finite AMPLRES.

Goodman and Silvestri [9] discuss a method of phase quantisation that is equivalent to PHASERES. They prove that phase quantisation in the Fourier domain causes degradation in the resulting inverse FT, in general. In particular they show that the step function can not be perfectly reconstructed from its phase discretised FT, thus we need infinite PHASERES to represent its FT. □

All theorems in this section are a worst case analysis. The previous theorem tells us that applying standard complexity theory to analyse continuous FTs is pointless. Obviously the result is of little relevance to CSMs that do not use the FT, or only use approximations of the FT. Typically in optical setups it will be the case that at some point of the computation, discretisations are introduced.

Theorem 2. *(h/v & DYRANGE) Let either* $\widehat{c} = h$ *or* $\widehat{c} = v$. *Then* $R_{\mathrm{D}_{T+1}} = \infty$.

Proof. Take the constant image $a = 1$. The horizontal FT $h(a)$ has value 0 everywhere except at $x = 0$ where it is a δ function, for all y. Hence there is no finite minimum DYRANGE that bounds the value at $h(a(0,y))$. A similar argument holds for v, the only difference is that we get the δ function at $v(a(x,0))$. □

It is worthwhile noting that restrictions on images (e.g. finite SPATIALRES) enable us to use Rayleigh's theorem [3, page 112] to specify a finite upper bound on the DYRANGE of $h(a)$ in terms of the complexity of image a [18].

Lemma 1. *Let* $z \in \mathbb{C}$, $\mu \in \mathbb{N}^+$ *and given* P *from Equation* (2), *then*

$$P(z,\mu) \in \left\{ z' \;\middle|\; z' = |z| \exp\left(\mathrm{i}\mu' \frac{2\pi}{\mu}\right), \mu' \in \{1, 2, \ldots, \mu\} \right\}\,.$$

Proof. Let $j = \left\lfloor \arg(z)\frac{\mu}{2\pi} + \frac{1}{2} \right\rfloor$, hence $j \in \mathbb{Z}$. Let $\arg(z)$ have range $0 < \arg(z) \leqslant 2\pi$. By substituting for $\arg(z)$ in j it is clear that $j \in \mu' \cup \{0\}$. Since we are working in radians, $j = 0$ gives the same value in P as $j = \mu$, hence $j \in \mu'$. □

Theorem 3. *($*$ & PHASERES) Let $\widehat{c} = *$. Then $R_{\mathrm{P}_{T+1}} = R_{\mathrm{P}_T}$.*

Proof (Sketch). We give a proof for the non-trivial case of finite PHASERES. The $*$ operation affects only image a. By Lemma 1 the set of phase angles in $\mathrm{range}(a)$ at TIME T is of the form $\{\Theta \mid \Theta = \mu'(2\pi/\mu)\}$. Our notation is in radians hence $n\Theta$, for all $n \in \mathbb{Z}$, is in the above set of μ angles. For the case of complex conjugation, $n = -1$. Thus the set of possible phases in $\mathrm{range}(a)$ at TIME $T+1$ is a subset of the set of phases in $\mathrm{range}(a)$ at TIME T. For details see [18]. □

Theorem 4. *($\cdot$ & AMPLRES) Let $\widehat{c} = \cdot$, then $R_{\mathrm{A}_{T+1}} = {R_{\mathrm{A}_T}}^2$.*

Proof. We give a proof for the non-trivial case of finite AMPLRES. The $\cdot$ operation affects only image a. For any $x, y \in [0, 1)$, let $z_a = \mathrm{range}(a(x, y))$, $z_b = \mathrm{range}(b(x, y))$. At TIME $T + 1$, $a(x, y)$ is replaced with $a'(x, y) = z_a z_b = |z_a||z_b| \exp(\mathrm{i}(\arg(a(x, y)) + \arg(b(x, y))))$. Let $R_{\mathrm{A}_T} = \mu$ in Equation (1), the values in a and b at TIME T are of the form

$$A(z, \mu) = \left\lfloor |z|\,\mu + \frac{1}{2} \right\rfloor \frac{1}{\mu} \exp(\mathrm{i} \times \arg(z))\,.$$

We are interested only in AMPLRES so we ignore the phase term. At TIME $T + 1$

$$|A(z_a, \mu)||A(z_b, \mu)| = \left\lfloor |z_a|\,\mu + \frac{1}{2} \right\rfloor \left\lfloor |z_b|\,\mu + \frac{1}{2} \right\rfloor \frac{1}{\mu^2}\,.$$

We are proving the theorem for the case that AMPLRES is finite, hence we know that at TIME T, $|z_a|$ and $|z_b|$ are rationals, moreover they are of the form $|z_a| = n/\mu$ and $|z_b| = m/\mu$ for some $n, m \in \mathbb{N}$. By substitution we simplify the above expression to get $|A(z_a, \mu)||A(z_b, \mu)| = nm/\mu^2$ In the worst case we require AMPLRES $\mu^2 = {R_{\mathrm{A}_T}}^2$ to represent the values in image a at TIME $T + 1$. □

Theorem 5. *($\cdot$ & DYRANGE) Let $\widehat{c} = \cdot$, then $R_{\mathrm{D}_{T+1}} = {R_{\mathrm{D}_T}}^2$.*

Proof. We give a proof for the non-trivial case of finite DYRANGE. The $\cdot$ operation affects only image a. Let z_a and z_b be defined as above. If $|z_a| = |z_b| = R_{\mathrm{D}_T}$ then at TIME $T+1$ we get the worst case of $R_{\mathrm{D}_{T+1}} = |a(x, y)| = {R_{\mathrm{D}_T}}^2$. It is easy to see that for all other values of $|z_a|$ and $|z_b|$, $R_{\mathrm{D}_{T+1}} < {R_{\mathrm{D}_T}}^2$. □

Unlike AMPLRES and DYRANGE, PHASERES is unaffected by multiplication:

Theorem 6. *($\cdot$ & PHASERES) Let $\widehat{c} = \cdot$, then $R_{\mathrm{P}_{T+1}} = R_{\mathrm{P}_T}$.*

Proof. We give a proof for the non-trivial case of finite PHASERES. The operation $\cdot$ affects only image a. We will show that the set of possible phases in $\mathrm{range}(a)$ at time $T + 1$ is a subset of the possible phases in $\mathrm{range}(a) \cup \mathrm{range}(b)$ at time T. For some (x, y) let $z_a = a(x, y)$ and $z_b = b(x, y)$. By definition $\cdot(z_a, z_b) =$

$|z_a||z_b|\exp(\mathrm{i}(\arg z_a + \arg z_b))$. At TIME T (from Equation (2)), $\arg z_a = \frac{n}{R_{\mathrm{P}_T}} 2\pi$ and $\arg z_b = \frac{m}{R_{\mathrm{P}_T}} 2\pi$, where $n, m \in \mathbb{N}$. Thus $\cdot(z_a, z_b) = |z_a||z_b|\exp\left(\mathrm{i}\left(\frac{n+m}{R_{\mathrm{P}_T}}\right)2\pi\right)$ which is in the set of possible phase values in range$(a) \cup$ range(b) at TIME T. $\square$

Theorem 7. *(+ & AMPLRES) Let $\widehat{c} = +$. Then $R_{\mathrm{A}_{T+1}} = \infty$.*

Proof. Suppose $R_{\mathrm{A}_T} = 1$ and $R_{\mathrm{P}_T} = 4$, then let $a(x, y) = \mathrm{i}$ and $b(x, y) = 1$. After the $+$ operation, at time $T + 1$, image a has value $a(x, y) = 1 + i = \sqrt{2}e^{\mathrm{i}\frac{1}{4}\pi}$. The CSM requires ∞ AMPLRES to represent the amplitude value $\sqrt{2}$. $\square$

If we restrict PHASERES to be 1 or 2 then we don't meet the worst case scenario described in the previous theorem, we introduce this restriction in Section 5.

Theorem 8. *(+ & DYRANGE) Let $\widehat{c} = +$. Then $R_{\mathrm{D}_{T+1}} = 2R_{\mathrm{D}_T}$.*

Proof. The operation $+$ has no effect on SPATIALRES hence without loss of generality we assume that a and b are everywhere constant, $a(x, y) = z_a = r_a e^{\mathrm{i}\Theta_a 2\pi}$ and $b(x, y) = z_b = r_b e^{\mathrm{i}\Theta_b 2\pi}$. Let $z_a = z_b$ and $|z_a| = R_{\mathrm{D}_T}$, in this case $z_a + z_b = 2z_a = 2r_a e^{\mathrm{i}\Theta_a 2\pi}$ and hence $R_{\mathrm{D}_{T+1}} = 2R_{\mathrm{D}_T}$. In fact this is the worst case since adding any pair of complex values that lie on the origin-centred disk of radius R_{D_T} gives a new complex value on the origin-centred disk of radius $2R_{\mathrm{D}_T}$. $\square$

Theorem 9. *(+ & PHASERES) Let $\widehat{c} = +$. Then $R_{\mathrm{P}_{T+1}} = \infty$.*

Proof. We give a proof for the non-trivial case of finite PHASERES. The operation $+$ has no effect on SPATIALRES hence without loss of generality we assume that a and b are everywhere constant. Let $a = 2$ and $b = \mathrm{i} = e^{\mathrm{i}\frac{1}{2}\pi}$. At time $T + 1$, $a = \sqrt{5}\exp\left(\mathrm{i}\tan^{-1}(1/2)\,\pi\right)$. Niven [14], Corollary 3.12, shows that $(\tan^{-1}(1/2))/\pi$ is irrational, thus we require infinite PHASERES for addition. $\square$

Theorem 10. *(st/ld/ρ & GRID) Let either $\widehat{c} = st$, $\widehat{c} = ld$ or $\widehat{c} = \rho$. Then there is no upper bound on the value of G_{T+1}.*

Proof. The address *decoding* function $\mathfrak{E}^{-1} : \mathcal{N} \to \mathbb{N}$ is Turing machine decidable. This is the only specific restriction on $\mathfrak{E}^{-1}$. Thus there is no upper bound on the natural number that an address parameter of *st* maps to. After a *st* operation we cannot bound GRID in terms of G_T, or any other complexity measure. The same argument holds for *ld* and ρ. $\square$

The previous theorem highlights the caveat of *reasonableness* in the definition of $\mathfrak{E}$ in Section 2. When we defined $\mathfrak{E}$ we did not wish to restrict the CSM programmer from coming up with a novel $\mathfrak{E}$ suited to her needs. However, for reasonable addressing functions we should expect the growth rate of $\mathfrak{E}^{-1}$, with respect to the ordering on $\mathcal{N}$, to be reasonable. For example, in Section 5 we restrict $\mathfrak{E}$ to being logspace Turing machine computable, which is an agreed notion of reasonableness in parallel complexity theory. As one can imagine, a complicated $\mathfrak{E}$ will leave lots of headaches for the optical engineer who has to implement it. Not only that, we would also have an incomplete complexity analysis of the CSM in question (unless of course we work the growth rate of $\mathfrak{E}$ into our analysis). The same remark applies to the next theorem.

Theorem 11. *(ld & spatialRes/freq) Let $\widehat{c} = ld$. Then there is no upper bound on the value of $R_{s_{T+1}}$ nor ν_T.*

Proof. After a *ld* operation with parameters $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{N}$, image a has spatialRes $R_{s_{T+1}} = R_{s_T}(\xi_2 - \xi_1 + 1)(\eta_2 - \eta_1 + 1)$. From Theorem 10 there is no upper bound on the growth of $\mathfrak{E}^{-1}$. Thus there is no upper bound on the *ld* parameters. After a *st* operation there is no upper bound on spatialRes in terms of R_{s_T}, or any other complexity measure. Analogously, for freq the upper bound is in terms of $\mathfrak{E}^{-1}$ rather than any of the complexity measures. □

If we have agreed upon a reasonable (bound on) $\mathfrak{E}$, then it is straightforward to derive an upper bound on spatialRes and freq at time $T + 1$.

Even though *br* has address parameters, the previous arguments do not apply.

Theorem 12. *(br & grid/freq) Let $\widehat{c} = br$. Then $G_{T+1} = G_T$.*

Proof. From the definition of a CSM configuration [19] the control must always be inside the initial (time 1) grid. Branching outside the current grid will always result in an undefined computation, hence *br* does not increase grid. □

5 $\mathcal{C}_2$-CSM

Motivated by a desire to apply standard complexity theory tools to the model, we define a restricted class of CSM.

Definition 11 ($\mathcal{C}_2$-CSM). *A $\mathcal{C}_2$-CSM is a CSM whose computation* time *is defined for $t \in \{1, 2, \ldots, T(n)\}$ and has the following restrictions:*

- *For all* time *t both* amplRes *and* phaseRes *have constant value of 2.*
- *For all* time *t each of* grid, spatialRes *and* dyRange *is $O(2^t)$ and* space *is redefined to be the product of all complexity measures except* time *and* freq.
- *Operations h and v compute the discrete FT (DFT) in the horizontal and vertical directions respectively.*
- *Given some* reasonable *binary word representation of the set of addresses $\mathcal{N}$, the address encoding function $\mathfrak{E} : \mathbb{N} \to \mathcal{N}$ is decidable by a logspace Turing machine.*

We have replaced the FT with the DFT [3]. freq is now solely dependent on spatialRes (rescaling the Fourier spectrum by changing freq is no longer necessary); hence freq is not an interesting complexity measure for $\mathcal{C}_2$-CSMs. The DFT is defined over a ring. Since dyRange is bounded, and amplRes and phaseRes are constant, we satisfy the definition of the DFT. The space restriction is not unique to our model, such restrictions can be found in [15, 7].

In Sect. 2 we stated that address encoding functions should be Turing machine computable, here we strengthen this condition. At first glance sequential logspace computability may perhaps seem like a strong restriction, but in fact it is quite weak. From an optical implementation point of view it should be the case that $\mathfrak{E}$ is not complicated, otherwise we cannot assume fast addressing. Other (sequential/parallel) models usually have a very restricted 'addressing function': in most cases it is simply the identity function on $\mathbb{N}$. Without an explicit or implicit restriction on the computational complexity of $\mathfrak{E}$, finding non-trivial upper bounds on the power of $\mathcal{C}_2$-CSMs is impossible as $\mathfrak{E}$ could encode an arbitrarily complex halting Turing machine. As a weaker restriction we could give a specific $\mathfrak{E}$. However, this restricts the generality of the model and prohibits the programmer from developing novel, reasonable, addressing schemes.

A $\mathcal{C}_2$-CSM resource usage table would contain no "∞" or "unbounded" entries. In further work we will give exact characterisations of the power of this model.

6 Discussion

We have analysed the growth of CSM complexity measures with respect to its operations over TIME. Table 1 shows that many variations on the CSM can not be analysed if we restrict ourselves to the standard tools from complexity theory.

The results in this paper are independent of any particular data representations or program restrictions. If we restrict ourselves to certain (continuous or discrete) data representations then clearly we change the properties of computations and can reduce the upper bounds on resource growth. Another way to restrict the model is to place restrictions on the syntactic structure of programs.

Earlier CSM versions [11, 12, 13, 19] used constant GRID. The function $\mathfrak{E}$ allows GRID to be a more useful complexity resource (see [18] for further remarks).

Table 1 describes growth in complexity if inputs are finite. The irrational values that give rise to the infinities in Table 1 are computable reals (say, in the sense of [17]). It would be interesting to analyse this aspect of the model by making use of results from the framework of real recursive function theory [10, 4, 5] or other approaches to analog or real computation [2, 17]. There has been little work towards a parallel complexity theory for analog computation, this would be interesting future work.

The results from this paper are not only interesting from a computational complexity viewpoint, but from a physical viewpoint also. For example Goodman [9] studies PHASERES in the same way we do, and is motivated by practical concerns (reconstructing digital holograms). The $\mathcal{C}_2$-CSM is more realistic than the CSM in terms of optical implementation; many optical information processing devices are pixellated (e.g. liquid-crystal displays and digital cameras) and operate over a finite set of grey levels [8]. Positive and negative rationals are routinely represented in optical architectures [6]. Clearly, the SPACE limitation decreases the difficulty of implementation.

This work is a starting point for developing CSM restrictions; in particular we defined the $\mathcal{C}_2$-CSM. Any restriction will exhibit resource growth less than or equal to that given by Table 1. Interesting future work would be to characterise the power of such restrictions. In further publications we will exactly characterise standard sequential and parallel complexity classes in terms of the $\mathcal{C}_2$-CSM.

For example, we will show that the $\mathcal{C}_2$-CSM satisfies the parallel computation thesis [1, 7, 15] and that the class NC is characterised in terms of the $\mathcal{C}_2$-CSM [18].

Acknowledgements

We thank Tom Naughton for many fruitful discussions and in particular for his collaboration on the CSM definition. The first author is funded by the Irish Research Council for Science, Engineering and Technology.

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